For \( c \) satisfying \( 0 \leq c \leq 1 \) it follows that
\[
\int_{1-c}^{1} x^n \, dx = \int_{0}^{1} x^n \, dx - \int_{0}^{1-c} x^n \, dx
= A_n[1 - (1 - c)^{n+1}]
= A_n[(n + 1)c - \frac{n(n + 1)}{2}c^2 + \cdots + (-1)^n c^{n+1}]. \tag{1}
\]

But by reflecting in the line \( x = 1/2 \), we also obtain
\[
\int_{1-c}^{1} x^n \, dx = \int_{c}^{0} (1 - x)^n \, dx
= \int_{0}^{c} 1 - nx + \frac{n(n - 1)}{2}x^2 - \cdots + (-1)^n x^n \, dx
= cA_0 - nc^2 A_1 + \frac{n(n - 1)}{2}c^3 A_2 - \cdots + (-1)^n c^{n+1} A_n. \tag{2}
\]

Since the two polynomials (1) and (2) in \( c \) agree for all \( c \) in \([0, 1]\), they must be identical. Comparing their linear terms gives the required result \( A_n = 1/(n + 1) \).  

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**REFERENCES**

where
\[ D_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \]

for \( n \) in \( \mathbb{N} \). Write
\[ r_n = \gamma - D_n. \]

R. M. Young [1] gave the following estimate for \( r_n \):
\[ \frac{1}{2(n+1)} < r_n < \frac{1}{2n}. \] \hspace{1cm} (1)

\[ \tilde{D}_n = \sum_{k=1}^{n} \frac{1}{k} - \log \left( n + \frac{1}{2} \right) \]
in place of \( D_n \) and showed that
\[ \frac{7}{960} \cdot \frac{1}{(n+1)^4} < \gamma - \tilde{D}_n + \frac{1}{24(n+\frac{1}{2})^2} < \frac{7}{960n^4}. \]

An earlier discussion of \( D_n \) can be found in Rippon [3]. Furthermore, DeTemple and Wang [4] established an estimate for \( r_n \) in which Bernoulli numbers are involved.

In this note we use an elementary method to give an exact representation of \( r_n \), from which asymptotic estimates for \( r_n \) are then derived. Our method is to calculate sums by means of integrals.

2. THE METHOD. Rewrite \( D_n \) as
\[ D_n = \sum_{k=1}^{n} \left( \frac{1}{k} - \int_{k}^{k+1} \frac{1}{x} \, dx \right) = \sum_{k=1}^{n} \int_{0}^{1} \frac{t}{k(k+t)} \, dt. \] \hspace{1cm} (2)

From (2) we obtain
\[ r_n = \sum_{k=n+1}^{\infty} \int_{0}^{1} \frac{t}{k(k+t)} \, dt \]
\[ = \sum_{k=n+1}^{\infty} \int_{0}^{1} t \left( \frac{1}{k(k+t)} - \frac{1}{k(k+1)} \right) \, dt + \int_{0}^{1} t \, dt \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \]
\[ = \sum_{k=n+1}^{\infty} \int_{0}^{1} \frac{t(1-t)}{k(k+1)(k+t)} \, dt + \frac{1}{n+1} \int_{0}^{1} t \, dt. \]

Write
\[ r_1(n) = \sum_{k=n+1}^{\infty} \int_{0}^{1} \frac{t(1-t)}{k(k+1)(k+t)} \, dt, \quad a_1 = \int_{0}^{1} t \, dt. \] \hspace{1cm} (3)
Then

\[ r_n = r_1(n) + \frac{a_1}{n+1}. \]  

(4)

Moreover,

\[ r_1(n) = \sum_{k=n+1}^{\infty} \int_0^1 \frac{t(1-t)}{k(k+1)(k+t)} \, dt \]

\[ = \sum_{k=n+1}^{\infty} \int_0^1 t(1-t) \left( \frac{1}{k(k+1)(k+t)} - \frac{1}{k(k+1)(k+2)} \right) \, dt \]

\[ + \sum_{k=n+1}^{\infty} \int_0^1 \frac{t(1-t)(2-t)}{k(k+1)(k+2)(k+t)} \, dt \]

\[ = \sum_{k=n+1}^{\infty} \int_0^1 \frac{t(1-t)(2-t)}{k(k+1)(k+2)(k+t)} \, dt + \frac{1}{2} \int_0^1 t(1-t) \, dt \]

\[ \frac{1}{(n+1)(n+2)}. \]

Let

\[ r_2(n) = \sum_{k=n+1}^{\infty} \int_0^1 \frac{t(1-t)(2-t)}{k(k+1)(k+2)(k+t)} \, dt, \quad a_2 = \frac{1}{2} \int_0^1 t(1-t) \, dt. \]  

(5)

Then

\[ r_n = r_2(n) + \frac{a_1}{n+1} + \frac{a_2}{(n+1)(n+2)}. \]

For \( m \) in \( \mathbb{N} \) with \( m \geq 2 \) we have

\[ r_m(n) = \sum_{k=n+1}^{\infty} \int_0^1 \frac{t(1-t)(2-t)\cdots(m-t)}{k(k+1)(k+2)\cdots(k+m)(k+t)} \, dt, \]

\[ a_m = \frac{1}{m} \int_0^1 t(1-t)\cdots(m-1-t) \, dt. \]  

(6)

By induction we get

\[ r_n = \sum_{k=1}^{m} \frac{a_k}{(n+1)(n+2)\cdots(n+k)} + r_m(n). \]

From (3) and (5) we learn that

\[ 2a_2 \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^2} < r_1(n) < 2a_2 \sum_{k=n+1}^{\infty} \frac{1}{k^2(k+1)}. \]
from which we derive (using the fact that \(a_2 = 1/12\)) the estimate

\[
\frac{1}{12(n+1)(n+2)} < r_1(n) < \frac{1}{12n(n+1)}.
\]  

(7)

Hence we arrive via (4) and (7) at

\[
\frac{1}{2(n+1)} + \frac{1}{12(n+1)(n+2)} < r_n < \frac{1}{2(n+1)} + \frac{1}{12n(n+1)}.
\]

which is stronger than (1).

From (6) we obtain (for \(m \geq 2\)):

\[
r_m(n) < a_{m+1} \sum_{k=n+1}^{\infty} \left( \frac{1}{(k-1)k \cdots (k+m-1)} - \frac{1}{k(k+1) \cdots (k+m)} \right)
\]

\[= \frac{a_{m+1}(n-1)!}{(n+m)!}\]

and

\[
r_m(n) > \sum_{k=n+1}^{\infty} \frac{(m+1)a_{m+1}(k-1)!}{(k+1)(k+m)!} > \frac{n(m+1)a_{m+1}}{n+2} \sum_{k=n+1}^{\infty} \frac{(k-2)!}{(k+m)!}.
\]

Since

\[
\sum_{k=n+1}^{\infty} \frac{(k-2)!}{(k+m)!} = \sum_{k=n+1}^{\infty} \frac{1}{(k-1)k \cdots (k+m)}
\]

\[= \sum_{k=n+1}^{\infty} \frac{1}{(m+1)}
\]

\[\times \left( \frac{1}{(k-1)k \cdots (k+m-1)} - \frac{1}{k(k+1) \cdots (k+m)} \right)
\]

\[= \frac{(n-1)!}{(m+1)(n+m)!}\]

we have

\[
r_m(n) > \frac{a_{m+1}n!}{(n+2)(n+m)!}.
\]

On the other hand, it is obvious from (6) that for \(m \geq 2\)

\[
\frac{1}{6m} (m-2)! \leq a_m \leq \frac{1}{6m} (m-1)!. \]

We conclude that

\[
\frac{1}{6(n+2)m(m+1)(m+n)} \leq r_m(n) \leq \frac{1}{6n(m+1)(m+n)}
\]  

(8)
for \( m \geq 2 \), where
\[
\binom{m+n}{m} = \frac{m! \cdot n!}{(m+n)!}.
\]

Taking into account (7), we see that (8) is also valid for \( m = 1 \). From (8) we infer
\[
\lim_{m \to \infty} r_m(n) = 0.
\]

We have thus established the following theorem:

**Theorem.** Let \( D_n = \sum_{k=1}^{n} k^{-1} - \log(n+1) \) and let \( \gamma = \lim_{n \to \infty} D_n \) be Euler’s constant. Then
\[
r_n = \gamma - D_n = \sum_{k=1}^{\infty} \frac{a_k}{(n+1) \cdots (n+k)},
\]

where
\[
a_1 = \frac{1}{2}, \quad a_k = \frac{1}{k} \int_{0}^{1} t(1-t) \cdots (k-1-t) \, dt \quad (k > 1).
\]

Furthermore,
\[
\frac{1}{6n(m+1)(m+n)} < r_n - \sum_{k=1}^{m} \frac{a_k}{(n+1) \cdots (n+k)} < \frac{1}{6n(m+1)(m+n)}.
\]

The referee kindly produced the following table for the numbers \( a_1, a_2, \ldots, a_8 \):

\[
a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{12}, \quad a_3 = \frac{1}{12}, \quad a_4 = \frac{19}{120}, \quad a_5 = \frac{9}{20}, \quad a_6 = \frac{863}{504}, \quad a_7 = \frac{1375}{168}, \quad a_8 = \frac{33953}{720}.
\]

He also pointed out that \( a_k \) can be expressed in terms of Stirling numbers of the first kind \( s(k, j) \) as
\[
a_k = \frac{(-1)^{k+1}}{k} \sum_{j=1}^{k} s(k, j) j + 1.
\]

Our proof is a completely elementary calculation applying integrals to estimate certain sums. The method can be applied to other cases as well.

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1. INTRODUCTION. One knows that $\log(1 - z) = -\sum_{n=1}^{\infty} z^n / n$ for $|z| \leq 1$ and $z \neq 1$. Because $1 - e^{i\theta} = -2i \sin(\theta/2) \cdot e^{i\theta/2}$ and $-i = e^{-\pi i/2}$, we see that

$$\log(1 - e^{i\theta}) = \log(-i) + \log \left(2 \sin \frac{\theta}{2}\right) + \log e^{i\theta/2} = \log \left(2 \sin \frac{\theta}{2}\right) - i \left(\frac{\pi}{2} - \theta\right),$$

and on taking real and imaginary parts of $-\log(1 - z)$ with $z = e^{i\theta} = \cos \theta + i \sin \theta$, it follows that

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\log \left(2 \sin \frac{\theta}{2}\right)$$

and

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi}{2} - \frac{\theta}{2}$$

for $0 < \theta < 2\pi$.

The relevant rule of thumb is that power series can safely be treated as if they were polynomials of [very] high degree provided one stays well away from the boundary of the disc of convergence. So, guessing that $\log(1 - e^{i\theta})$ has imaginary part $\sum_{n \geq 1} (\sin n\theta)/n = (\pi - \theta)/2$ for $0 < \theta < 2\pi$ is scary stuff requiring the presence of a qualified mathematician.* Do not try it at home.

In fact, oops! What if $\theta$ creeps down to zero? Surely, all the terms of the series become zero? But its purported sum becomes $\pi/2$!

2. EVALUATION OF AN INTEGRAL. Not to worry. Look carefully at the graph of $(\sin x)/x$. It’s the sine curve wriggling pathetically as it is squeezed between the hyperbolae $xy = 1$ and $xy = -1$.

* MGR: Mathematical Guidance Recommended. Possible use of strong technical language and presence of naked singularities.